9.0 INTRODUCTION

In Chapter 1 we treated the propagation of electromagnetic waves in anisotropic media. It was shown how the properties of the propagating wave can be determined from the dielectric tensor \( \varepsilon_{ij} \) or more conveniently from the index ellipsoid. In this chapter we consider the problem of propagation of optical radiation in crystals in the presence of an applied electric field. We find that in certain types of crystals it is possible to effect a change in the index of refraction that is proportional to the applied electric field. This is the linear electro-optic effect (also known as the Pockels effect, named after F. Pockels who studied the effect in 1893). It affords a convenient and widely used means of controlling the intensity or phase of the propagating radiation. This modulation is used in an ever expanding number of applications, including the impressing of information onto optical beams, active mode locking of lasers for generation of ultrashort optical pulses, and optical beam deflection. Some of these applications will be discussed further in this chapter. Modulation and deflection of laser beams by acoustic beams are also considered later in this chapter.

9.1 LINEAR ELECTRO-OPTIC EFFECT

The propagation of optical radiation in a crystal can be described completely in terms of the dielectric tensor. In Chapter 1 we found that, given a direction of propagation in a crystal, in general two possible linearly polarized modes exist—the so-called normal modes of propagation. Each mode possesses a unique direction of polarization (i.e., direction of \( \mathbf{D} \)) and a corresponding index of refraction (i.e., a velocity of propagation). The mutually orthogonal polarization directions (\( \mathbf{D} \) vectors) and the indices of the two modes are found most conveniently by using the index ellipsoid, which assumes its simplest form in the principal coordinate system:

\[
\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1
\]  

(9.1-1)
where the directions \(x, y, \) and \(z\) are the principal dielectric axes—that is, the directions in the crystal along which the \(D\) and \(E\) field vectors are parallel. We note that \(\frac{1}{n_x^2}, \frac{1}{n_y^2}, \) and \(\frac{1}{n_z^2}\) are the principal values of the impermeability tensor, which is defined as

\[
\eta = \frac{\varepsilon_0}{\varepsilon} \tag{9.1-2}
\]

where \(\varepsilon\) is the dielectric tensor.

According to the quantum theory of solids, the optical dielectric tensor depends on the distribution of charges in the crystal. The application of an applied electric field can cause a redistribution of the charges and possibly a slight deformation of the crystal lattice. The net result is a change in the optical impermeability tensor. This known as the electro-optic effect. The linear electro-optic coefficients are defined traditionally as

\[
\Delta \eta_{ij} = \eta(E) - \eta(0) \equiv \Delta \left( \frac{1}{n^2} \right)_{ij} \equiv r_{jk}E_k \tag{9.1-3}
\]

where \(E\) is the applied electric field, \(E_k\) is the \(k\) component \((k = x, y, z)\) of the electric field, and the summation over repeated indices is assumed. In the above equation, we neglected the high-order terms. The constants \(r_{jk}\) are the linear electro-optic coefficients. For convenience, we use the convention \(1 = x, 2 = y, 3 = z\).

Thus the index ellipsoid of a crystal in the presence of an applied electric field is given by

\[
\eta_{ij}(E)x_ix_j = 1 \tag{9.1-4}
\]

or equivalently,

\[
\left( \frac{1}{n_x^2} + r_{1k}E_k \right)x^2 + \left( \frac{1}{n_y^2} + r_{2k}E_k \right)y^2 + \left( \frac{1}{n_z^2} + r_{3k}E_k \right)z^2 + 2xyr_{12k}E_k + 2yzr_{23k}E_k + 2zx\eta_{3k}E_k = 1 \tag{9.1-5}
\]

where \(E_k\) is the \(k\) component \((k = x, y, z)\) of the electric field and the summation over repeated indices is assumed. In Equation (9.1-5), we have used the symmetry property of the impermeability tensor \((\eta_{ij} = \eta_{ji})\). As a result of the "mixed terms" in Equation (9.1-5), the principal axes \((x, y, z)\) of the crystal are no longer the principal axes of the index ellipsoid.

The impermeability tensor is symmetric provided that the medium is lossless and optically inactive (i.e., no optical activity or optical rotatory power). In this case the linear electro-optic coefficients satisfy the following symmetric relationship:

\[
r_{ijk} = r_{jik} \tag{9.1-6}
\]

Because of the symmetry, it is convenient to introduce contracted indices to abbreviate the notation. They are defined as

\[
1 = (11) = (xx) \\
2 = (22) = (yy) \\
3 = (33) = (zz) \\
4 = (23) = (32) = (yz) = (zy) \\
5 = (12) = (21) = (xz) = (zx) \\
6 = (13) = (31) = (xy) = (yx)
\]
Using these contracted indices, we can write

\[ r_{1k} = r_{11k} \]
\[ r_{2k} = r_{22k} \]
\[ r_{3k} = r_{33k} \quad (k = 1, 2, 3) \quad \text{(9.1.8)} \]
\[ r_{4k} = r_{23k} = r_{32k} \]
\[ r_{5k} = r_{31k} = r_{13k} \]
\[ r_{6k} = r_{12k} = r_{21k} \]

It is important to remember that the contraction of the indices is just a matter of convenience. These matrix elements \((6 \times 3)\) do not have the usual tensor transformation or multiplication properties. The permutation symmetry reduces the number of independent elements of \(r_{jk}\) from 27 to 18.

According to Equation (9.1-5), the normal modes of propagation and the corresponding refractive indices associated with the modes in the presence of an applied electric field will depend on the magnitude as well as the direction of the applied field. Using the method of the index ellipsoid, the refractive indices of the normal modes of propagation can be obtained. The linear electro-optic effect is the change in the refractive indices of the normal modes of propagation (e.g., ordinary and extraordinary modes in uniaxial crystals) that is caused by and is proportional to an applied electric field. This effect exists only in crystals that do not possess inversion symmetry (centrosymmetric crystals). The structure of these crystals remains invariant under the inversion operation (a coordinate transformation by replacing \(\mathbf{r}\) with \(-\mathbf{r}\)).

This statement can be justified as follows. Assume that in a crystal possessing an inversion symmetry, the application of an electric field \(E\) along some direction causes a change \(\Delta n_1 = sE\) in the index, where \(s\) is a constant characterizing the linear electro-optic effect. If the direction of the field is reversed, the change in the index is given by \(\Delta n_2 = s(-E)\), but because of the inversion symmetry the two directions are physically equivalent, so \(\Delta n_1 = -\Delta n_2\). This requires that \(s = -s\), which is possible only for \(s = 0\), so no linear electro-optic effect can exist in centrosymmetric crystals. The division of all crystal classes into those that do and those that do not possess an inversion symmetry is an elementary consideration in crystallography and this information is widely tabulated [1].

Using the contracted notation, the index ellipsoid in the presence of the applied electric field can be written

\[
\left[ \frac{1}{n_1^2} + \Delta \left( \frac{1}{n_1^2} \right) \right] x^2 + \left[ \frac{1}{n_2^2} + \Delta \left( \frac{1}{n_2^2} \right) \right] y^2 + \left[ \frac{1}{n_3^2} + \Delta \left( \frac{1}{n_3^2} \right) \right] z^2 \\
+ 2yz \Delta \left( \frac{1}{n_4^2} \right) + 2zx \Delta \left( \frac{1}{n_5^2} \right) + 2xy \Delta \left( \frac{1}{n_6^2} \right) = 1
\]

or equivalently,

\[
\left( \frac{1}{n_1^2} + r_{1k} E_k \right) x^2 + \left( \frac{1}{n_2^2} + r_{2k} E_k \right) y^2 + \left( \frac{1}{n_3^2} + r_{3k} E_k \right) z^2 \\
+ 2yz r_{4k} E_k + 2zx r_{5k} E_k + 2xy r_{6k} E_k = 1
\]

(9.1-9)

where, again, summations over repeated indices are assumed.

Furthermore, the linear change in the coefficients
\[
\left( \frac{1}{n^2} \right)_c, \quad l = 1, \ldots, 6
\]

due to an arbitrary dc electric field \( \mathbf{E}(E_x, E_y, E_z) \) can be expressed in a matrix form as

\[
\begin{bmatrix}
\Delta(1/n^2)_1 \\
\Delta(1/n^2)_2 \\
\Delta(1/n^2)_3 \\
\Delta(1/n^2)_4 \\
\Delta(1/n^2)_5 \\
\Delta(1/n^2)_6
\end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\
r_{21} & r_{22} & r_{23} \\
r_{31} & r_{32} & r_{33} \\
r_{41} & r_{42} & r_{43} \\
r_{51} & r_{52} & r_{53} \\
r_{61} & r_{62} & r_{63} \end{bmatrix} \begin{bmatrix} E_x \\
E_y \\
E_z \end{bmatrix}
\]

(9.1-10)

where, using the rules for matrix multiplication, we have, for example,

\[
\Delta \left( \frac{1}{n^2} \right)_c = -r_{b1}E_x + r_{b2}E_y + r_{b3}E_z
\]

(9.1-11)

The 6 \times 3 matrix with elements \( r_{ij} \) is called the electro-optic tensor. We have shown above that in crystals possessing an inversion symmetry (centrosymmetric), \( r_{ij} = 0 \). The form, but not the magnitude, of the tensor \( r_{ij} \) can be derived from symmetry considerations [1], which dictate which of the 18 \( r_{ij} \) coefficients are zero, as well as the relationships that exist between the remaining coefficients. In Table 9.1 we give the form of the electro optic tensor for all the noncentro symmetric crystal classes. The electro-optic coefficients of some crystals are given in Table 9.2.

**TABLE 9.1** Electro-optic Coefficients in Contracted Notation for all Crystal Symmetry Classes

| Centrosymmetric: \( \{1,2/m,mmm,4/m,mmm, 3, 3m, 6/m, 6/mmmm, m3, m3m\} \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| \[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\] |

| Triclinic: \( \{1\} \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| \[
\begin{bmatrix}
r_{11} & r_{12} & r_{13} \\
r_{21} & r_{22} & r_{23} \\
r_{31} & r_{32} & r_{33} \\
r_{41} & r_{42} & r_{43} \\
r_{51} & r_{52} & r_{53} \\
r_{61} & r_{62} & r_{63} \\
\end{bmatrix}
\] |

| Monoclinic: \( \{2 \parallel x_2 \} \) | \( \{2 \parallel x_3 \} \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \[
\begin{bmatrix}
0 & r_{12} & 0 \\
0 & r_{22} & 0 \\
0 & r_{32} & 0 \\
r_{41} & r_{42} & r_{43} \\
r_{51} & r_{52} & r_{53} \\
r_{61} & r_{62} & r_{63} \\
\end{bmatrix}
\] | \[
\begin{bmatrix}
0 & 0 & r_{13} \\
0 & 0 & r_{23} \\
0 & 0 & r_{33} \\
r_{41} & r_{42} & r_{43} \\
r_{51} & r_{52} & r_{53} \\
r_{61} & r_{62} & r_{63} \\
\end{bmatrix}
\] |
TABLE 9.1 (cont’d)

<table>
<thead>
<tr>
<th>$m (m \perp x_3)$</th>
<th>$m (m \perp x_3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[\eta_1 \ 0 \ \eta_3]$</td>
<td>$[\eta_1 \ \eta_2 \ 0]$</td>
</tr>
<tr>
<td>$\eta_3 \ 0 \ \eta_3$</td>
<td>$\eta_1 \ \eta_2 \ 0$</td>
</tr>
<tr>
<td>$\eta_3 \ 0 \ \eta_3$</td>
<td>$\eta_1 \ \eta_2 \ 0$</td>
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<tr>
<td>$0 \ \eta_3 \ 0$</td>
<td>$0 \ \eta_3 \ 0$</td>
</tr>
<tr>
<td>$0 \ \eta_3 \ 0$</td>
<td>$0 \ \eta_3 \ 0$</td>
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<tr>
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<td>$\eta_3 \ 0 \ \eta_3$</td>
</tr>
<tr>
<td>$0 \ \eta_3 \ 0$</td>
<td>$0 \ \eta_3 \ 0$</td>
</tr>
<tr>
<td>Orthorhombic:</td>
<td>Orthorhombic:</td>
</tr>
<tr>
<td>$mmm$</td>
<td>$2mm$</td>
</tr>
<tr>
<td>$[0 \ 0 \ 0]$</td>
<td>$[0 \ 0 \ \eta_3]$</td>
</tr>
<tr>
<td>$0 \ 0 \ 0$</td>
<td>$0 \ 0 \ \eta_2$</td>
</tr>
<tr>
<td>$0 \ 0 \ 0$</td>
<td>$0 \ 0 \ \eta_3$</td>
</tr>
<tr>
<td>$\eta_4 \ 0 \ 0$</td>
<td>$\eta_4 \ 0 \ \eta_2$</td>
</tr>
<tr>
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<td>$\eta_3 \ 0 \ \eta_2$</td>
</tr>
<tr>
<td>$0 \ \eta_3 \ 0$</td>
<td>$0 \ \eta_3 \ 0$</td>
</tr>
<tr>
<td>Tetragonal:</td>
<td>Tetragonal:</td>
</tr>
<tr>
<td>$4$</td>
<td>$4\bar{4}$</td>
</tr>
<tr>
<td>$[0 \ 0 \ \eta_3]$</td>
<td>$[0 \ 0 \ \eta_3]$</td>
</tr>
<tr>
<td>$0 \ 0 \ \eta_3$</td>
<td>$0 \ 0 \ -\eta_3$</td>
</tr>
<tr>
<td>$0 \ 0 \ \eta_3$</td>
<td>$0 \ 0 \ 0$</td>
</tr>
<tr>
<td>$\eta_4 \ \eta_3 \ 0$</td>
<td>$\eta_4 \ \eta_3 \ 0$</td>
</tr>
<tr>
<td>$\eta_3 \ -\eta_4 \ 0$</td>
<td>$\eta_3 \ -\eta_4 \ 0$</td>
</tr>
<tr>
<td>$0 \ 0 \ 0$</td>
<td>$0 \ 0 \ \eta_3$</td>
</tr>
<tr>
<td>$4mm$</td>
<td>$42m (2 \perp x_1)$</td>
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<td>$[0 \ 0 \ \eta_3]$</td>
<td>$[0 \ 0 \ 0]$</td>
</tr>
<tr>
<td>$0 \ 0 \ \eta_3$</td>
<td>$0 \ 0 \ 0$</td>
</tr>
<tr>
<td>$0 \ 0 \ \eta_3$</td>
<td>$0 \ 0 \ 0$</td>
</tr>
<tr>
<td>$0 \ \eta_3 \ 0$</td>
<td>$\eta_4 \ 0 \ 0$</td>
</tr>
<tr>
<td>$\eta_3 \ 0 \ 0$</td>
<td>$\eta_4 \ 0 \ 0$</td>
</tr>
<tr>
<td>$0 \ 0 \ 0$</td>
<td>$0 \ 0 \ \eta_3$</td>
</tr>
<tr>
<td>Trigonal:</td>
<td>Trigonal:</td>
</tr>
<tr>
<td>$3$</td>
<td>$32$</td>
</tr>
<tr>
<td>$[\eta_1 \ -\eta_2 \ \eta_3]$</td>
<td>$[\eta_1 \ 0 \ 0]$</td>
</tr>
<tr>
<td>$-\eta_1 \ \eta_2 \ \eta_3$</td>
<td>$\eta_1 \ 0 \ 0$</td>
</tr>
<tr>
<td>$0 \ 0 \ \eta_3$</td>
<td>$0 \ 0 \ \eta_3$</td>
</tr>
<tr>
<td>$\eta_4 \ \eta_3 \ 0$</td>
<td>$\eta_4 \ \eta_3 \ 0$</td>
</tr>
<tr>
<td>$\eta_3 \ -\eta_4 \ 0$</td>
<td>$\eta_3 \ -\eta_4 \ 0$</td>
</tr>
<tr>
<td>$-\eta_2 \ -\eta_1 \ 0$</td>
<td>$0 \ -\eta_4 \ 0$</td>
</tr>
<tr>
<td>$3m (m \perp x_1)$</td>
<td>$3m (m \perp x_2)$</td>
</tr>
<tr>
<td>$[0 \ -\eta_2 \ \eta_3]$</td>
<td>$[\eta_1 \ 0 \ \eta_3]$</td>
</tr>
<tr>
<td>$0 \ \eta_2 \ \eta_3$</td>
<td>$\eta_1 \ 0 \ \eta_3$</td>
</tr>
<tr>
<td>$0 \ 0 \ \eta_3$</td>
<td>$0 \ \eta_3 \ \eta_3$</td>
</tr>
<tr>
<td>$0 \ \eta_3 \ 0$</td>
<td>$0 \ \eta_3 \ 0$</td>
</tr>
<tr>
<td>$\eta_3 \ 0 \ 0$</td>
<td>$\eta_5 \ 0 \ 0$</td>
</tr>
<tr>
<td>$-\eta_2 \ 0 \ 0$</td>
<td>$0 \ -\eta_3 \ 0$</td>
</tr>
</tbody>
</table>
**TABLE 9.1 (cont’d)**

<table>
<thead>
<tr>
<th>Hexagonal:</th>
<th>6</th>
<th>bmm</th>
<th>622</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \begin{bmatrix} 0 &amp; 0 &amp; \eta_{13} \ 0 &amp; 0 &amp; \eta_{13} \ 0 &amp; 0 &amp; \eta_{33} \ r_{41} &amp; r_{51} &amp; 0 \ r_{51} &amp; r_{41} &amp; 0 \ 0 &amp; 0 &amp; 0 \end{bmatrix} )</td>
<td>( \begin{bmatrix} 0 &amp; 0 &amp; \eta_{13} \ 0 &amp; 0 &amp; \eta_{13} \ 0 &amp; 0 &amp; \eta_{33} \ 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{bmatrix} )</td>
<td>( \begin{bmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{bmatrix} )</td>
<td></td>
</tr>
<tr>
<td>( \hat{\eta} )</td>
<td>( \hat{\eta} )</td>
<td>( \hat{\eta} )</td>
<td></td>
</tr>
</tbody>
</table>

\( \begin{bmatrix} \eta_{11} & -\eta_{22} & 0 \\ \eta_{11} & \eta_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -r_{22} & -\eta_{11} & 0 \end{bmatrix} \) | \( \begin{bmatrix} 0 & -r_{22} & 0 \\ 0 & r_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \) | \( \begin{bmatrix} \eta_{11} & 0 & 0 \\ 0 & \eta_{11} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \) |

\( \hat{\eta}m^2 \) (m \( \perp \) \( r_1 \)) | \( \hat{\eta}m^2 \) (m \( \perp \) \( r_2 \)) |

**Cubic:**

<table>
<thead>
<tr>
<th>( \bar{4}3m, \ 23 )</th>
<th>( \bar{4}3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \begin{bmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \ r_{41} &amp; 0 &amp; 0 \ 0 &amp; r_{41} &amp; 0 \ 0 &amp; 0 &amp; r_{41} \end{bmatrix} )</td>
<td>( \begin{bmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{bmatrix} )</td>
</tr>
</tbody>
</table>

* The symbol over each matrix is the conventional symmetry-group designation.

**TABLE 9.2** Linear Electro-optic Coefficients of Some Commonly Used Crystals

<table>
<thead>
<tr>
<th>Substance</th>
<th>Symmetry</th>
<th>Wavelength ( \lambda ) (( \mu m ))</th>
<th>Electro-optic Coefficients ( r_{ik} (10^{-12} \text{ m/V}) )</th>
<th>Index of Refraction ( n_i ) ( (10^{11} \text{ m/V}) )</th>
<th>( n^3 r )</th>
<th>Dielectric Constant ( \varepsilon )</th>
</tr>
</thead>
<tbody>
<tr>
<td>CdTe</td>
<td>( \bar{4}3m )</td>
<td>1.0</td>
<td>( (T) \ r_{41} = 4.5 )</td>
<td>( n = 2.84 )</td>
<td>103</td>
<td>( (S) \ \varepsilon = 9.4 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3.39</td>
<td>( (T) \ r_{41} = 6.8 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>10.6</td>
<td>( (T) \ r_{41} = 6.8 )</td>
<td>( n = 2.60 )</td>
<td>120</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>23.35</td>
<td>( (T) \ r_{41} = 5.47 )</td>
<td>( n = 2.58 )</td>
<td>94</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>27.95</td>
<td>( (T) \ r_{41} = 5.04 )</td>
<td>( n = 2.53 )</td>
<td>82</td>
<td></td>
</tr>
<tr>
<td>GaAs</td>
<td>( \bar{4}3m )</td>
<td>0.9</td>
<td>( r_{41} = 1.1 )</td>
<td>( n = 3.60 )</td>
<td>51</td>
<td>( (S) \ \varepsilon = 13.2 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.15</td>
<td>( (T) \ r_{41} = 1.43 )</td>
<td>( n = 3.42 )</td>
<td>58</td>
<td>( (T) \ \varepsilon = 12.3 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3.39</td>
<td>( (T) \ r_{41} = 1.24 )</td>
<td>( n = 3.33 )</td>
<td>45</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>10.6</td>
<td>( (T) \ r_{41} = 1.51 )</td>
<td>( n = 3.33 )</td>
<td>54</td>
<td></td>
</tr>
<tr>
<td>GaP</td>
<td>( \bar{4}3m )</td>
<td>0.55–1.3</td>
<td>( (T) \ r_{41} = -1.0 )</td>
<td>( n = 3.66–3.08 )</td>
<td>( (S) \ \varepsilon = 10 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.633</td>
<td>( (S) \ r_{41} = 0.97 )</td>
<td>( n = 3.32 )</td>
<td>35</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.15</td>
<td>( (S) \ r_{41} = -1.10 )</td>
<td>( n = 3.10 )</td>
<td>33</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>3.39</td>
<td>( (S) \ r_{41} = -0.97 )</td>
<td>( n = 3.02 )</td>
<td>27</td>
<td></td>
</tr>
<tr>
<td>( \beta )-ZnS (sphalerite)</td>
<td>( \bar{4}3m )</td>
<td>0.4</td>
<td>( (T) \ r_{41} = 1.1 )</td>
<td>( n = 2.52 )</td>
<td>18</td>
<td>( (T) \ \varepsilon = 16 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.5</td>
<td>( (T) \ r_{41} = 1.81 )</td>
<td>( n = 2.42 )</td>
<td>( (S) \ \varepsilon = 12.5 )</td>
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<tr>
<td></td>
<td></td>
<td>0.6</td>
<td>( (T) \ r_{41} = 2.1 )</td>
<td>( n = 2.36 )</td>
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<td>( (S) \ r_{41} = -1.6 )</td>
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<td>( (S) \ r_{41} = -1.4 )</td>
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<td>Substance</td>
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<td>Electro-optic Coefficients ($r_{ik}$ $(10^{-12}$ m/V))</td>
<td>Index of Refraction ($n_i$)</td>
<td>$n_i^2$ $(10^{-12}$ m/V)</td>
<td>Dielectric Constant ($\varepsilon_i(\varepsilon_0)$)</td>
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<tr>
<td>ZnSe</td>
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<td>0.548</td>
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<td>Pb$<em>{0.10}$La$</em>{0.64}$</td>
<td>$\infty$</td>
<td>0.546</td>
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<td>$n_o = 3.019$</td>
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<td>LiNbO$_3$</td>
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<td>$n_o = 2.286$</td>
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<td>($T_c = 1730$ °C)</td>
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<td>($T$) $r_{33} = 30.9$</td>
<td>$n_o = 2.073$</td>
<td>($S$) $\varepsilon_3 = 41$</td>
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<td>($T$) $r_{31} = 32.6$</td>
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<td>($S$) $r_{33} = 4.5$</td>
<td>$n_e = 2.180$</td>
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<td>($S$) $r_{31} = 15$</td>
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<td>($S$) $r_{33} = 23$</td>
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<td>($S$) $r_{33} = 27$</td>
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<td>($T$) $\varepsilon_1 = 51$</td>
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<td>($S$) $r_{33} = 4.5$</td>
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<td>($T$) $\varepsilon_3 = 45$</td>
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<td>($S$) $r_{31} = 15$</td>
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<td>($S$) $\varepsilon_3 = 41$</td>
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<td>($S$) $r_{33} = 23$</td>
<td>$n_e = 2.065$</td>
<td>($S$) $\varepsilon_3 = 43$</td>
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<tr>
<td>AgGaS$_2$</td>
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<td>($T$) $r_{41} = 4.0$</td>
<td>$n_o = 2.55$</td>
<td>($T$) $\varepsilon_1 = 51$</td>
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<td>($T$) $r_{63} = 3.0$</td>
<td>$n_e = 2.607$</td>
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<td>CsH$_2$AsO$_4$ (CDA)</td>
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<td>0.55</td>
<td>($T$) $r_{41} = 14.8$</td>
<td>$n_o = 1.572$</td>
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<td></td>
<td>($T$) $r_{63} = 18.2$</td>
<td>$n_e = 1.550$</td>
<td>($T$) $\varepsilon_3 = 45$</td>
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<td>KH$_2$PO$_4$ (KDP)</td>
<td>42m</td>
<td>0.546</td>
<td>($T$) $r_{41} = 8.77$</td>
<td>$n_o = 1.5115$</td>
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<td>$n_e = 1.4698$</td>
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<td>0.633</td>
<td>($T$) $r_{41} = 8$</td>
<td>$n_o = 1.2074$</td>
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<td>($T$) $r_{63} = 9.7$</td>
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<td>($T$) $n_o r_{63} = 33$</td>
<td>$n_e = 2.553$</td>
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### TABLE 9.2 (cont’d)

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<tr>
<th>Substance</th>
<th>Symmetry</th>
<th>Wavelength ( \lambda ) ((\mu m))</th>
<th>Electro-optic Coefficients ( n_k ) ( (10^{12} \text{ m/V}) )</th>
<th>Index of Refraction ( n_r ) ( (10^{12} \text{ m/V}) )</th>
<th>Dielectric Constant* ( \varepsilon_r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>KD(_2)PO(_4) (KD(^+)P)</td>
<td>42(m)</td>
<td>0.546</td>
<td>0.053</td>
<td>1.5079</td>
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<td>1.502</td>
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<td>1.462</td>
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<tr>
<td>(NH(_4))(_2)HPO(_4) (ADP)</td>
<td>42(m)</td>
<td>0.546</td>
<td>0.053</td>
<td>1.5266</td>
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<td>0.633</td>
<td>0.088</td>
<td>1.4808</td>
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<td>0.241</td>
<td>1.5220</td>
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<td>(NH(_4))(_3)D(_2)PO(_4) (AD(^+)P)</td>
<td>42(m)</td>
<td>0.633</td>
<td>0.1640</td>
<td>1.5116</td>
<td>3600</td>
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<td>0.1640</td>
<td>1.475</td>
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<tr>
<td>BaTiO(_3) ( (T_c = 395 \text{ K}) )</td>
<td>4(mm)</td>
<td>0.546</td>
<td>0.1640</td>
<td>1.5476</td>
<td>135</td>
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<td>0.162</td>
<td>1.536</td>
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<tr>
<td>(KT(_2)O(_3)) ( \text{KTN} )</td>
<td>4(mm)</td>
<td>0.633</td>
<td>0.0800(T_c = 28 )</td>
<td>2.318</td>
<td>3000</td>
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<td>0.0800(T_c = 28 )</td>
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<td>0.0800(T_c = 16 )</td>
<td>2.281</td>
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<tr>
<td>Ba(<em>{0.25})K(</em>{0.75})Nb(_2)O(_6) ( (T_c = 205 \text{ K}) )</td>
<td>4(mm)</td>
<td>0.633</td>
<td>0.1640</td>
<td>2.3117</td>
<td>3400</td>
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<td>0.162</td>
<td>1.960</td>
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<tr>
<td>o-H(_2)O(_3) ( (\text{OH}) )</td>
<td>2(mm)</td>
<td>0.633</td>
<td>0.1640</td>
<td>2.280</td>
<td>15 \text{ MHz}</td>
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<td>0.162</td>
<td>2.329</td>
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<td></td>
<td></td>
<td>0.162</td>
<td>2.169</td>
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<tr>
<td>KNbO(_3) (2(mm) )</td>
<td>2(mm)</td>
<td>0.633</td>
<td>0.1640</td>
<td>2.280</td>
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<td>0.162</td>
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<td></td>
<td></td>
<td>0.162</td>
<td>2.169</td>
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</tbody>
</table>
| KIO\(_3\) | 1 | 0.500 | 0.66 | 1.700 | 1.828 | 1.832 |}

* \( T \) = low frequency from dc through audio range; \( S \) = high frequency.

---

### EXAMPLE: ELECTRO-OPTIC EFFECT IN KH\(_2\)PO\(_4\)

Consider the specific example of a crystal of potassium dihydrogen phosphate (KH\(_2\)PO\(_4\)), also known as KDP. The crystal has a fourfold axis of symmetry \((C_4)\), which by strict convention is taken as the \(z\) (optic) axis, as well as two mutually orthogonal twofold axes of symmetry that lie in the plane normal to \(z\). These are designated as the \(x\) and \(y\) axes. The symmetry group of this crystal is \(42\(m\)\). A crystal with an \(n\)-fold axis of symmetry \((C_n)\) is invariant under a rotation of \(2\pi/n\) around the axis. Using Table 9.1, we take the electro-optic tensor in the form of

\[
\tau_{ij} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\tau_{41} & 0 & 0 \\
0 & \tau_{41} & 0 \\
0 & 0 & \tau_{63}
\end{bmatrix}
\]

(9.1-12)
so the only nonvanishing elements are \( r_{41} - r_{52} \) and \( r_{53} \). Using Equation (9.1-9), we obtain the equation of the index ellipsoid in the presence of a field \( \mathbf{E}(E_x, E_y, E_z) \) as

\[
\frac{x^2}{n_0^2} + \frac{y^2}{n_0^2} + \frac{z^2}{n_e^2} + 2r_{41}E_x y z + 2r_{41}E_y x z + 2r_{53}E_z x y = 1 \tag{9.1-13}
\]

where the constants involved in the first three terms do not depend on the field and, since the crystal is uniaxial, are taken as \( n_c = n_y = n_z \), and \( n_e = n_x \). We thus find that the application of an electric field causes the appearance of "mixed" terms in the equation of the index ellipsoid. These are the terms with \( xy, xz, \) and \( yz \). This means that the major axes of the ellipsoid, with a field applied, are no longer parallel to the \( x, y, \) and \( z \) axes. It becomes necessary, then, to find the directions and magnitudes of the new axes, in the presence of \( \mathbf{E} \), so that we may determine the effect of the field on the propagation. To be specific, we choose the direction of the applied field parallel to the \( z \) axis, so Equation (9.1-13) becomes

\[
\frac{x^2}{n_0^2} + \frac{y^2}{n_0^2} + \frac{z^2}{n_e^2} + 2r_{53}E_z x y = 1 \tag{9.1-14}
\]

The problem is one of finding a new coordinate system—\( x', y', z' \)—in which the equation of the ellipsoid (9.1-14) contains no mixed terms; that is, it is of the form

\[
\frac{x'^2}{n_0'} + \frac{y'^2}{n_0'} + \frac{z'^2}{n_e'} = 1 \tag{9.1-15}
\]

\( x', y', \) and \( z' \) are then the directions of the major axes of the ellipsoid in the presence of an external field applied parallel to \( z \). The lengths of the major axes of the ellipsoid are, according to Equation (9.1-15), \( 2n_0', 2n_0', \) and \( 2n_e' \), and these will, in general, depend on the applied field.

In the case of (9.1-14) it is clear from inspection that in order to put the equation in diagonal form, we need to choose a coordinate system \( x', y', z' \) where \( z' \) is parallel to \( z \), and because of the symmetry of (9.1-14) in \( x \) and \( y \), \( x' \) and \( y' \) are related to \( x \) and \( y \) by a 45° rotation, as shown in Figure 9.1. The transformation relations from \( x, y \) to \( x', y' \) are thus

\[
x = x' \cos 45^\circ + y' \sin 45^\circ
\]

\[
y = -x' \sin 45^\circ + y' \cos 45^\circ
\]

which, upon substitution in Equation (9.1-14), yield

\[
\left( \frac{1}{n_0'} - \frac{\epsilon_{63}E_z}{n_0'} \right) x'^2 + \left( \frac{1}{n_0'} + \frac{\epsilon_{63}E_z}{n_0'} \right) y'^2 + \frac{z'^2}{n_e'} = 1 \tag{9.1-16}
\]

Equation (9.1-16) shows that \( x', y', \) and \( z \) are indeed the principal axes of the ellipsoid when a field is applied along the \( z \) direction. According to Equation (9.1-16), the length of the \( x' \) axis of the ellipsoid is \( 2n_0' \), where

\[
\frac{1}{n_0'} = \frac{1}{n_0'} - \epsilon_{63}E_z
\]

**Figure 9.1** Coordinate rotation to transform the quadratic equation into a diagonal form. In this case, the \( z \)-axis is the fourfold axis of symmetry, and \( x \) and \( y \) are the twofold axes of symmetry in the crystals with 42\( m \) symmetry.
which, assuming \( r_{63}E_z \ll n_0^{-2} \) and using the differential relation \( dn = -(n^3/2) d(1/n^3) \), gives for the change in \( n_x \).

\[
n_x = n_0 + \frac{n_0^3}{2} r_{63}E_z \quad (9.1-17)
\]

and, similarly,

\[
n_y = n_0 - \frac{n_0^3}{2} r_{63}E_z \quad (9.1-18)
\]

\[
n_z = n_0 = n_0 \quad (9.1-19)
\]

Let us now consider the case when the applied electric field is parallel to the \( x \) axis, so that Equation (9.1-13) becomes

\[
\frac{x^2}{n_0^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} + 2\varepsilon_0\varepsilon_n E_x z = 1 \quad (9.1-20)
\]

In this case it is clear from inspection of Equation (9.1-20) that the new principal axis \( x' \) will coincide with the \( x \) axis, because the “mixed” term involves only \( y \) and \( z \). A rotation in the \( yz \) plane is therefore required to put it in diagonal form. Let \( \theta \) be the angle between the new coordinate \( y'z' \) and the old coordinate \( yz \). The transformation from \( x, y, z \) to \( x', y', z' \) is given by

\[
\begin{align*}
x &= x' \\
y &= y' \cos \theta - z' \sin \theta \\
z &= y' \sin \theta + z' \cos \theta
\end{align*} \quad (9.1-21)
\]

We now substitute Equation (9.1-21) for \( y \) and \( z \) in Equation (9.1-20) and require that the coefficient for the \( y'z' \) term vanishes. This yields

\[
\frac{x'^2}{n_0^2} + \left( \frac{1}{n_y} + r_{41} E_x \tan \theta \right) y'^2 + \left( \frac{1}{n_z} - r_{41} E_x \tan \theta \right) z'^2 = 1 \quad (9.1-22)
\]

with \( \theta \) given by

\[
\tan \theta = \frac{2n_0^2 n_y^2}{n_z - n_0^2} r_{41} E_x \quad (9.1-23)
\]

The new index ellipsoid (9.1-22) has its principal axes rotated at an angle \( \theta \) about the \( x \) axis with respect to the principal axes of the crystal when a field \( E_x \) is applied along the \( x \) axis. This angle is very small, even for a moderately high field. For KDP with an applied field of \( E_x = 10^6 \) V/m, this angle is only 0.04°. According to Equation (9.1-23), this angle is only significant for materials with \( n_0 = n_0 \). In particular, \( \theta = 45^\circ \) when \( n_0 = n_0 \). The new principal refractive indices, according to Equation (9.1-22), are given by

\[
\begin{align*}
n_{x'} &= n_x \\
n_{y'} &= n_0 - \frac{1}{2} n_0^3 r_{41} E_x \tan \theta \\
n_{z'} &= n_0 + \frac{1}{2} n_0^3 r_{41} E_x \tan \theta
\end{align*} \quad (9.1-24)
\]

For KDP with a moderate field \( E_x \), \( \theta \) is small and is almost linearly proportional to \( r_{41} E_x \) according to Equation (9.1-23). Therefore the change in the refractive indices is of second order in \( E_x \).
EXAMPLE: ELECTRO-OPTIC EFFECT IN LiNbO$_3$

The LiNbO$_3$ crystal has a crystal symmetry of $3m$. The electro-optic coefficients are in the form, according to Table 9.1,

\[
\begin{bmatrix}
0 & -n_2 & n_3 \\
0 & n_2 & 0 \\
0 & 0 & n_3 \\
\end{bmatrix}
\]

(9.1-25)

We now consider the case when the applied electric field is along the $c$ axis ($z$ axis) of the crystal so that the equation of the index ellipsoid can be written, according to Equation (9.1-9),

\[
\left(\frac{x^2}{n_o^2} + \frac{y^2}{n_e^2} + \frac{z^2}{n_3^2}\right) = 1
\]

(9.1-26)

where $n_o$ and $n_e$ are the ordinary and extraordinary refractive indices, respectively. Since no mixed terms appear in Equation (9.1-26), the principal axes of the new index ellipsoid remain unchanged. The lengths of the new semi-axes are

\[
\begin{align*}
n_x &= n_o - \frac{1}{2} n_o^3 r_{13} E \\
n_y &= n_o - \frac{1}{2} n_e^3 r_{13} E \\
n_z &= n_e - \frac{1}{2} n_e^3 r_{33} E
\end{align*}
\]

(9.1-27)

Note that under the influence of an applied electric field in the direction of the $c$ axis, the crystal remains uniaxially anisotropic. If a beam of light is propagating along the $x$ axis, the birefringence seen by it is

\[
n_z - n_y = (n_e - n_o) - \frac{1}{2} (n_o^3 r_{33} - n_e^3 r_{13}) E
\]

(9.1-28)

Note that the birefringence can be tuned electrically.

The electro-optic effect in the important $43m$ crystal class (GaAs, InP, ZnS) is treated in detail in Appendix F.

The General Solution

We now consider the problem of optical propagation in a crystal in the presence of an external dc field along an arbitrary direction. The index ellipsoid with the dc field on is given by Equation (9.1-4), which is written in the quadratic form

\[
\eta_{ij} x_i x_j = 1
\]

(9.1-29)

where $\eta_{ij}$ is the impermeability tensor in the presence of the applied electric field. We also use the convention of summation over repeated indices. Our problem consists of finding the directions and magnitudes of the principal axes of the ellipsoid (9.1-29). This is often achieved by a rotation of the coordinate system. In the new coordinate system, the quadratic from (9.1-29) becomes
\[ \eta_{11}x_1^2 + \eta_{22}x_2^2 + \eta_{33}x_3^2 = 1 \]  
(9.1-30)

The axes of the new coordinate system are the principal axes of the ellipsoid, and the lengths of the principal axes are \(2\sqrt{1/\eta_{11}}, 2\sqrt{1/\eta_{22}}, 2\sqrt{1/\eta_{33}}\).

The mathematical process of transforming a quadratic form (9.1-29) into its principal form, where all mixed terms disappear, is equivalent to the diagonalization of the matrix \(\eta_{ij}\):

\[
\eta_{ij} = \begin{bmatrix}
\eta_{11} & \eta_{12} & \eta_{13} \\
\eta_{21} & \eta_{22} & \eta_{23} \\
\eta_{31} & \eta_{32} & \eta_{33}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{n_x^2} + r_{1k}E_k & r_{2k}E_k & r_{3k}E_k \\
r_{6k}E_k & \frac{1}{n_y^2} + r_{4k}E_k & r_{4k}E_k \\
r_{5k}E_k & r_{4k}E_k & \frac{1}{n_z^2} + r_{5k}E_k
\end{bmatrix}
\]  
(9.1-31)

where summations over repeated indices \(k\) are assumed. The diagonalization of the permeability tensor (or matrix) is often done by solving the following eigenvalue problem:

\[
\begin{bmatrix}
\eta_{11} & \eta_{12} & \eta_{13} \\
\eta_{21} & \eta_{22} & \eta_{23} \\
\eta_{31} & \eta_{32} & \eta_{33}
\end{bmatrix} \begin{bmatrix}
V \end{bmatrix} = \eta \begin{bmatrix}
V \end{bmatrix}  
\]  
(9.1-32)

For a real symmetric matrix (or tensor), the above equation yields three real eigenvectors and three real eigenvalues. The eigenvalues are exactly \(\eta'_{11}, \eta'_{22}, \text{ and } \eta'_{33}\), and the eigenvectors are parallel to the principal axes of the ellipsoid.

**EXAMPLE: ELECTRO-OPTIC EFFECT IN KH₂PO₄**

To illustrate the method of matrix diagonalization, we use the example of KH₂PO₄ (KDP) with a dc field along the crystal \(z\) axis, which was solved earlier in a somewhat less formal fashion.

The index ellipsoid is given by Equation (9.1-14) as

\[
\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} + 2r_{63}E_{xy} = 1
\]  
(9.1-33)

The permeability matrix is thus

\[
\eta_{ij} = \begin{bmatrix}
\frac{1}{n_x^2} & \eta_{63}E_z & 0 \\
\eta_{63}E_z & \frac{1}{n_y^2} & 0 \\
0 & 0 & \frac{1}{n_z^2}
\end{bmatrix}
\]  
(9.1-34)

The eigenvalues are given according to Equation (9.1-32) as the roots of the equation

\[
\begin{vmatrix}
\frac{1}{n_x^2} - \eta & \eta_{63}E_z & 0 \\
\eta_{63}E_z & \frac{1}{n_y^2} - \eta & 0 \\
0 & 0 & \frac{1}{n_z^2} - \eta
\end{vmatrix} = 0
\]  
(9.1-35)
which upon evaluation is

\[
\left(\frac{1}{n_e^2} - \eta\right)\left(\frac{1}{n_e^2} - \eta\right)^2 - (n_0E_z)^2 = 0
\]  
\[\text{(9.1-36)}\]

The roots are

\[
\eta_{11} = \frac{1}{n_e^2}
\]
\[
\eta_{12} = \frac{1}{n_e^2} + \epsilon_3 E_z
\]
\[
\eta_{13} = \frac{1}{n_e^2} - \epsilon_3 E_z
\]  
\[\text{(9.1-37)}\]

in agreement with Equation (9.1-16). These roots are used, one at a time, in Equation (9.1-32) to obtain the eigenvectors (or the principal axes of the ellipsoid).

The linear electro-optic effect (Pockels effect) can also be written in terms of the change in the dielectric constant. Starting from the definition of the dielectric impermeability tensor of Equation (9.1-2), \(\eta\epsilon = \epsilon_0\), we take a differentiation and obtain

\[
(\Delta\eta)\epsilon + \eta(\Delta\epsilon) = 0
\]  
\[\text{(9.1-38)}\]

Multiplying both sides of Equation (9.1-38) by \(\epsilon\) from the left, we obtain

\[
\Delta\epsilon = -\frac{\epsilon(\Delta\eta)\epsilon}{\epsilon_0}
\]  
\[\text{(9.1-39)}\]

where \(\Delta\eta\) is the change of the dielectric impermeability tensor due to an applied electric field. In terms of tensor elements, the above equation can be written

\[
\Delta\epsilon_{ij} = \epsilon_{ij}(E) - \epsilon_{ij}(0) = -\sum_k \epsilon_0 n_i^2 n_j^2 \epsilon_{ijk} E_k
\]  
\[\text{(9.1-40)}\]

where \(n_i\) and \(n_j\) are the principal refractive indices for polarization along the principal axes \(i\) and \(j\), respectively.

### 9.2 ELECTRO-OPTIC MODULATION—PHASE, AMPLITUDE

We have demonstrated in the previous section that an applied electric field can change the index ellipsoid of certain crystals. We also know that the propagation characteristics are governed by the index ellipsoid. Consequently, we can employ the electro-optic effect of these crystals to manipulate the propagation of optical waves, fundamentally their phase and polarization state. This modulation can, if desired, be converted into amplitude modulation. As an example, let us consider the propagation of a beam of polarized light in a KDP crystal under the influence of an applied electric field. Figure 9.2 shows a schematic drawing of a \(c\)-cut KDP crystal plate (crystal cut with surface perpendicular to the \(c\) axis) with an electric field \(\mathbf{E}\) applied parallel to the \(z\) axis (\(z\) axis is parallel to \(c\) axis).

The new principal axes \((x', y', z)\) for KDP with \(\mathbf{E}\) applied parallel to \(z\) are shown in Figure 9.2. If we consider propagation along the \(z\) direction, then according to the procedure described in Chapter 1, we need to determine the ellipse formed by the intersection of the
plane \( z = 0 \) (in general, the plane that contains the origin and is normal to the propagation direction) and the index ellipsoid. The equation of this ellipse is obtained from Equation (9.1-16) by putting \( z = 0 \) and is

\[
\left( \frac{1}{n_0^2} - r_{63}E_z \right)x'^2 + \left( \frac{1}{n_0^2} + r_{63}E_z \right)y'^2 = 1
\]  

(9.2-1)

It follows from the index ellipsoid method that the two normal modes of propagation are polarized along \( x' \) and \( y' \) and that their indices of refraction are \( n_{x'} \) and \( n_{y'} \), which are given by Equations (9.1-17) and (9.1-18).

We are now in a position to take up the concept of electro-optic phase retardation. We consider an optical field that is incident normally on the \( x'y' \) plane. For this propagation, the birefringence is given by

\[
n_{x'} - n_{y'} = n_0^3r_{63}E
\]

(9.2-2)

Let \( l \) be the thickness of the KDP plate. The phase retardation of this plate is then given by

\[
\Gamma = \frac{\omega}{c}(n_{x'} - n_{y'})l = \frac{2\pi}{\lambda}(n_{x'} - n_{y'})l = \frac{2\pi}{\lambda}n_0^3r_{63}El = \frac{2\pi}{\lambda}n_0^3r_{63}V
\]

where \( V \) is the voltage applied and is given by \( V = Fl \). We know from our discussion in Chapter 1 that phase retardation plates are polarization-state converters. Here we have a plate with a phase retardation proportional to the applied voltage. Consequently, we are able to convert the polarization state of the incident beam of light into a desired polarization state electrically. To illustrate this, let us assume that the incident beam of light is linearly polarized with its \( E_{\text{in}} \) vector along the \( x \) direction. The polarization state at the input plane (\( z = 0 \)) can be represented by a Jones vector (in the principal coordinate axes \( x', y' \)),

\[
E_{\text{in}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

(9.2-4)

The polarization state (in the principal coordinate axes \( x', y' \)) of the emerging beam of light at the output plane (\( z = d \)) is given by

\[
E_{\text{out}} = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\Gamma/2} \\ e^{i\Gamma/2} \end{bmatrix}
\]

(9.2-5)

where \( \Gamma \) is given by Equation (9.2-3). Figure 9.3 shows the polarization ellipse of the output beam of light at various values of the phase retardation \( \Gamma \). The voltage that yields a phase retardation of \( \pi \) is known as the half-wave voltage and is given in this case by
\[ V_\pi = \frac{\lambda}{2n_0/63} \]  

(9.2-6)

where \( \lambda \) is the wavelength of light. The half-wave voltage for a \( z \)-cut KDP plate at \( \lambda = 633 \) nm is about 9.3 kV. Note that the half-wave voltage is proportional to the wavelength and is inversely proportional to the relevant electro-optic coefficient. A phase retardation of \( \pi \) is needed to transform a polarization state into its orthogonal polarization state.

**Amplitude Modulation**

An examination of Figure 9.3 reveals that the electrically induced birefringence causes a wave launched at \( z = 0 \) with its polarization along the \( x \) axis to acquire a polarization component along the \( y \) axis, which grows with distance at the expense of the \( x \) component until, at a position at which \( \Gamma = \pi \), the polarization state becomes parallel to the \( y \) axis. If this position corresponds to the output plane of the crystal and if one inserts at this point a polarizer at right angles to the input polarization—that is, one that allows only \( E_x \) to pass—then with the field on, the optical beam passes through unattenuated, whereas with the field off (\( \Gamma = 0 \)), the output beam is blocked off completely by the crossed output polarizer. This control of the optical energy flow serves as the basis of the electro-optic amplitude modulation of light.

A typical arrangement of an electro-optic amplitude modulator is shown in Figure 9.4. It consists of an electro-optic crystal placed between two crossed polarizers, which, in turn, are at an angle of 45° with respect to the new (electrically induced) principal axes \( x' \) and \( y' \). To be specific, we show how this arrangement is achieved using a KDP crystal. Also included in the optical path is a naturally birefringent crystal that introduces a fixed retardation, so the total retardation \( \Gamma \) is the sum of the retardation due to this crystal and the electrically induced one.

The polarization state of the beam emerging from the crystal is given by Equation (9.2-5). With the presence of a polarizer along the \( y \) axis, the amplitude of the transmitted beam can be obtained by a simple geometric projection:

\[ E'_{\text{out}} = E_{\text{out}} \cdot \hat{y} \]  

(9.2-7)
Using
\[
\hat{y} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]

we obtain
\[
E'_{\text{out}} = -i \sin(\Gamma/2)
\]

The ratio of the output intensity to the input is thus
\[
\frac{I'_o}{I_i} = \sin^2 \left( \frac{\Gamma}{2} \right) = \sin^2 \left[ \frac{\pi}{2} \frac{V}{V_\pi} \right]
\]

The second equality in Equation (9.2-10) was obtained from Equation (9.2-6). The transmission factor \((I'_o/I_i)\) is plotted in Figure 9.5 against the applied voltage.

Figure 9.5 Transmission factor of a cross-polarized electro-optic modulator as a function of an applied voltage. The modulator is biased to the point \(\Gamma = \pi/2\), which results in a 50% intensity transmission. A small applied sinusoidal voltage modulates the transmitted intensity about the bias point.
The process of amplitude modulation of an optical signal is also illustrated in Figure 9.5. The modulator is usually biased with a fixed retardation $\Gamma = \pi/2$ to the 50% transmission point where the slope is maximum. This bias can be achieved by applying a voltage $V = V_r/2$ or, more conveniently, by using a naturally birefringent crystal as in Figure 9.4 to introduce a phase difference (retardation) of $\pi/2$ between the $x'$ and $y'$ components. A small sinusoidal modulation voltage would then cause a nearly sinusoidal modulation of the transmitted intensity as shown.

To treat the situation depicted by Figure 9.5 mathematically, we take

$$\Gamma = \frac{\pi}{2} + \Gamma_m \sin \omega_m t$$  \hspace{1cm} (9.2-11)

where the retardation bias is taken as $\pi/2$, and $\Gamma_m$ is related to the amplitude $V_m$ of the modulation voltage by Equation (9.2-6); thus $\Gamma_m = \pi(V_m/V_r)$.

Using Equation (9.2-10), we obtain

$$\frac{I_o}{I_i} = \sin^2 \left( \frac{\pi}{4} + \frac{\Gamma_m}{2} \sin \omega_m t \right)$$  \hspace{1cm} (9.2-12)

$$= \frac{1}{2} [1 + \sin(\Gamma_m \sin \omega_m t)]$$  \hspace{1cm} (9.2-13)

which, for $\Gamma_m \ll 1$, becomes

$$\frac{I_o}{I_i} \approx \frac{1}{2} (1 + \Gamma_m \sin \omega_m t)$$  \hspace{1cm} (9.2-14)

so that the intensity modulation is a linear replica of the modulating voltage $V_m \sin \omega_m t$. If the condition $\Gamma_m \ll 1$ is not fulfilled, it follows from Figure 9.4 or from Equation (9.2-12) that the intensity variation is distorted and will contain an appreciable amount of the higher (odd) harmonics. The dependence of the distortion on $\Gamma_m$ is discussed further in Problem 9.3.

In Figure 9.6 we show how some information signal $f(t)$ (the electric output of a phonograph stylus in this case) can be impressed electro-optically as an amplitude modulation on a

![Figure 9.6 An optical communication link using an electro-optic modulator.](image-url)
laser beam and subsequently be recovered by an optical detector. The details of the optical detection are considered in Chapter 11.

**Phase Modulation of Light**

In the preceding section we saw how the modulation of the state of polarization, from linear to elliptic, of an optical beam by means of the electro-optic effect can be converted, using polarizers, to intensity modulation. Here we consider the situation depicted by Figure 9.7, in which, instead of there being equal components along the induced birefringent axes (x' and y' in Figure 9.2), the incident beam is polarized parallel to one of them, x' say. In this case the application of the electric field does not change the state of polarization, but merely changes the output phase by

\[ \Delta \phi_{x'} = \frac{\omega}{c} \Delta n_{x'} l = \frac{2\pi}{\lambda} \Delta n_{x'} l \]  (9.2.15)

where \( l \) is the length of the crystal. From Equation (9.1.17),

\[ \Delta \phi_{x'} = \frac{\omega n_0^2 k_3}{2c} E_z l = \frac{\pi n_0^2 k_3}{\lambda} V = \frac{\pi V}{V_\pi} \]  (9.2.16)

If the applied voltage is sinusoidal and is taken as

\[ V = V_\pi \sin \omega_m t \]  (9.2.17)

then an incident optical field, which at the input (z = 0) face of the crystal is given by \( E_{\text{in}} = A \exp(i\omega t) \), will emerge as

\[ E_{\text{out}} = A \exp \left[ i \left( \omega t - \frac{\omega}{c} n_0 l - \frac{V_m}{V_\pi} \pi \sin \omega_m t \right) \right] \]  (9.2.18)

where \( l \) is the length of the crystal. Dropping the constant phase factor, which is of no consequence here, we rewrite the last equation as

\[ E_{\text{out}} = A \exp[i(\omega t - \delta \sin \omega_m t)] \]  (9.2.19)
where
\[
\delta = \frac{\omega n_{0}^{2} r_{63} E_{m} l}{2c} = \frac{n_{0}^{2} r_{63} V_{m}}{\lambda} = \frac{V_{m}}{V_{e}} \pi
\]  
(9.2-20)

is referred to as the phase modulation index. The optical field is thus phase modulated with a modulation index \( \delta \). If we use the Bessel function identity
\[
\exp(-i\delta \sin \omega_{m} t) = \sum_{m=-\infty}^{\infty} J_{n}(\delta) \exp(-i\omega_{m} t)
\]  
(9.2-21)

we can rewrite Equation (9.2-20) as
\[
E_{\text{out}} = A \sum_{n=-\infty}^{\infty} J_{n}(\delta) e^{(i\omega_{m} - n\omega_{0}) t}
\]  
(9.2-22)

which form gives the distribution of energy in the sidebands as a function of the modulation index \( \delta \). We note that, for \( \delta = 0 \), \( J_{0}(0) = 1 \) and \( J_{n}(\delta) = 0 \), \( n \neq 0 \). Another point of interest is that the phase modulation index \( \delta \) as given by Equation (9.2-20) is one-half the phase retardation \( \Gamma \) as given by Equation (9.2-3).

**Transverse Electro-optic Modulators**

In the examples of electro-optic phase retardation discussed earlier in this section, the electric field was applied along the direction of light propagation. This is the so-called longitudinal mode of modulation. A more desirable mode of operation is the transverse one, in which the field is applied perpendicular to the direction of propagation. The reason is that in this case the field electrodes do not interfere with the optical beam, and the retardation, being proportional to the product of the field times the crystal length, can be increased by the use of longer crystals. In the longitudinal case the retardation, according to Equation (9.2-3), is proportional to \( E_{y} l = V \) and is independent of the crystal length \( l \). Figures 9.1 and 9.2 suggest how transverse retardation can be obtained using a KDP crystal with the actual arrangement shown in Figure 9.8. The light propagates along \( y' \) and its polarization is in the \( x'z' \) plane at 45° from the \( z \) axis. The retardation, with a field applied along \( z \), is, from Equations (9.1-10) and (9.1-12),
\[
\Gamma = \phi_{z'} - \phi_{z} = \frac{\omega l}{c} \left( n_{o} - n_{e} \right) + \frac{n_{o}^{2}}{2} r_{63} \left( \frac{V}{d} \right)
\]  
(9.2-23)

**Figure 9.8** A transverse electro-optic amplitude modulator using a \( \text{KH}_{2}\text{PO}_{4} \) (KDP) crystal in which the field is applied normal to the direction of propagation.